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ON SOME RESULTS OF N. V. SMIRNOV CONCERNING LIMIT
 DISTRIBUTIONS FOR VARIATIONAL SERIES

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1. Introduction. Let ξ_1, \dots, ξ_n be n mutually independent and identically distributed random variables (rv). For every k ($1 \leq k \leq n$) denote by ξ_{kn} the rv that assumes the k th value in descending order of size among the values assumed by ξ_1, \dots, ξ_n . So, e.g., we have

$$\xi_{1n} = \max (\xi_1, \dots, \xi_n).$$

Many authors (cf. [1], [2], [11]) have investigated the asymptotic behavior of the distribution function (df) of the maximal term ξ_{1n} as $n \rightarrow \infty$. The most complete results, which may be said to summarize in a sense this series of investigations, were obtained by B. V. Gnedenko [3]. In particular, he determined the class of all df's which can be a limit of the df of the normalized maximal term $(\xi_{1n} - b_n)/a_n$ as $n \rightarrow \infty$, where $a_n > 0$ and b_n are suitably chosen real numbers.

Gnedenko's results were generalized by N. V. Smirnov [12]. He showed that the class of all proper limit distribution laws for the normalized rv ξ_{kn} consists of the following:

$$\begin{aligned} \Phi_\alpha(x; k) &= 0 && \text{if } x < 0, \\ &= \exp(-x^{-\alpha}) \sum_{s=0}^{k-1} x^{-s\alpha}/s! && \text{if } x \geq 0; \\ (1.1) \quad \Psi_\alpha(x; k) &= \exp(-|x|^\alpha) \sum_{s=0}^{k-1} |x|^{s\alpha}/s! && \text{if } x < 0, \\ &= 1 && \text{if } x \geq 0; \end{aligned}$$

where $\alpha > 0$, and

$$\Lambda(x; k) = \exp(-e^{-x}) \sum_{s=0}^{k-1} e^{-sx}/s!$$

The limit distributions for the maximal term are obtained by putting $k = 1$.

The variable ξ_{kn} is a well-defined function of the rv's ξ_1, \dots, ξ_n and the index k ($1 \leq k \leq n$)

$$(1.2) \quad \xi_{kn} = f(\xi_1, \dots, \xi_n; k)$$

which satisfies the identity

$$f(\xi_1, \dots, \xi_n; k) = -f(-\xi_1, \dots, -\xi_n; n - k + 1).$$

This relation permits us to carry over results found for the df of ξ_{kn} to the df of $\xi_{n-k+1, n}$ and conversely.

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Gnedenko's problem was generalized in another direction in [4] and [7]–[10], where the limit distributions of the maximal term ξ_{1n} were considered in the case where the initial rv's ξ_1, \dots, ξ_n are not necessarily identically distributed. It is the purpose of the present paper to generalize Smirnov's result in this direction.

2. Statement of the problem. Let ξ_1, \dots, ξ_n be mutually independent rv's and let

$$F_i(x) = P(\xi_i \leq x), \quad i = 1, \dots, n.$$

It is easy to see that for every k ($1 \leq k \leq n$)

$$(2.1) \quad \Phi_{kn}(x) = P(\xi_{kn} \leq x) = \prod_{i=1}^n F_i(x) + \sum_{s=1}^{k-1} \sum_{(n*s)} [\prod_{j \in (n*s)^c} F_j(x) \prod_{i \in (n*s)} (1 - F_i(x))]$$

where $(n*s)$ is any subset of the set $(1, \dots, n)$, consisting of s indices, $(n*s)^c$ —its complement and the inner summation is over all such subsets. Since

$$(2.2) \quad \Phi_{1n}(x) = \prod_{i=1}^n F_i(x)$$

then, whenever x is such that $F_i(x) > 0$ ($i = 1, \dots, n$), then the df $\Phi_{kn}(x)$ may be written also in the form

$$(2.3) \quad \Phi_{kn}(x) = \Phi_{1n}(x) \sum_{s=0}^{k-1} \sum \prod_{(n*s)} (1/F_i(x) - 1),$$

where for the sake of brevity we write $\sum \prod_{(n*s)}$ instead of $\sum_{(n*s)} \prod_{i \in (n*s)}$ and put $\sum \prod_{(n*0)} \equiv 1$.

THEOREM 2.1. *For every df $\Phi(x)$ and every k there exists a sequence of df's $F_i(x)$ such that*

$$(2.4) \quad \lim_{n \rightarrow \infty} \Phi_{kn}(x) = \Phi(x).$$

PROOF. Consider an auxiliary sequence of mutually independent rv's η_i with df's

$$(2.5) \quad G_i(x) = P(\eta_i \leq x) = \begin{cases} 0, & \text{if } x < 0, \\ = 1 - 2/(1 + e^{ix}), & \text{if } x \geq 0. \end{cases}$$

For arbitrary k and n ($1 \leq k \leq n$) let us denote

$$\eta_{kn} = f(\eta_1, \dots, \eta_n; k),$$

where the function f is the same as in (1.2), and let

$$(2.6) \quad \Gamma_{kn}(x) = P(\eta_{kn} \leq x).$$

It is clear that $\Gamma_{kn}(x) = 0$ if $x \leq 0$, and for $x > 0$, in view of (2.3), the df $\Gamma_{kn}(x)$ may be represented in the form

$$(2.7) \quad \Gamma_{kn}(x) = \sum_{s=0}^{k-1} T_{sn}(x),$$

where

$$(2.8) \quad T_{sn}(x) = 2^s \prod_{i=1}^n G_i(x) \sum \prod_{(n*s)} (e^{ix} - 1)^{-1}.$$

For fixed s and $a > 0$ consider the expression

$$\Delta_{mn}(x) = |T_{s,n+m}(x) - T_{sn}(x)|,$$

where $x \geq a$. We have

$$\begin{aligned} \Delta_{mn}(x) < 2^s [\prod_{i=n+1}^{n+m} (1 + 2/(e^{ix} - 1)) - 1] \sum \prod_{(n*s)} (e^{ix} - 1)^{-1} \\ + \sum \prod_{((n+m)*s)} (e^{ix} - 1)^{-1} - \sum \prod_{(n*s)} (e^{ix} - 1)^{-1}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \prod_{i=n+1}^{n+m} (1 + 2/(e^{ix} - 1)) &< \prod_{i=n+1}^{\infty} (1 + 2/(e^{ia} - 1)) < \exp(2 \sum_{i=n+1}^{\infty} (e^{ia} - 1)^{-1}) \\ &< \exp(4a^{-2} \sum_{i=n+1}^{\infty} i^{-2}), \\ \sum \prod_{(n*s)} (e^{ix} - 1)^{-1} &\leq (\sum_{i=1}^{\infty} (e^{ia} - 1)^{-1})^s \leq (2a^{-2} \sum_{i=1}^{\infty} i^{-2})^s, \end{aligned}$$

and for $s \geq 1$ we have

$$\begin{aligned} \sum \prod_{((n+m)*s)} (e^{ix} - 1)^{-1} - \sum \prod_{(n*s)} (e^{ix} - 1)^{-1} \\ < \sum_{i=1}^s [(\sum_{i=1}^n (e^{ix} - 1)^{-1})^{s-i} (\sum_{j=n+1}^{n+m} (e^{ix} - 1)^{-1})^i] \\ < s(2/a^2)^s (\sum_{i=1}^{\infty} i^{-2})^s (\sum_{j=n+1}^{\infty} j^{-2}). \end{aligned}$$

Thus we obtain

$$\Delta_{mn}(x) < C(a, s; n) = o(1), \quad (n \rightarrow \infty)$$

where $x \geq a$ and m are arbitrary. This estimate proves the existence of the limit

$$(2.9) \quad \lim_{n \rightarrow \infty} \Gamma_{kn}(x) = \Gamma(x)$$

and the uniform convergence of the sequence $\Gamma_{kn}(x)$ in any interval of the form $[a, \infty)$, where $a > 0$. It follows in particular that the limit function $\Gamma(x)$ is a distribution, i.e. $\Gamma(+\infty) = 1$.

We will now show that

$$(2.10) \quad \Gamma(0+) = 0.$$

Because of (2.5) and (2.8), we have for any positive x

$$T_{sn}(x) < (2^s \sum \prod_{(n*s)} e^{-ix}) / \prod_{j=1}^n (1 + e^{-jx}),$$

and since

$$\sum \prod_{(n*s)} e^{ix} < (\sum_{i=1}^n e^{-ix})^s < 1/(e^x - 1)^s,$$

we also get

$$T_{sn}(x) < 2^s (e^x - 1)^{-s} / \prod_{i=1}^n (1 + e^{-ix}).$$

Therefore, if $0 < x < \ln 2$, then by (2.7) we get

$$\Gamma(x) < k2^k(e^x - 1)^{-k} / \prod_{i=1}^{\infty} (1 + e^{-ix}).$$

It is easy to verify that $1 + \alpha > e^{\alpha/2}$ if $0 < \alpha < 1$. Hence if $x > 0$, we have

$$\prod_{i=1}^{\infty} (1 + e^{-ix}) > \exp(2^{-1} \sum_{i=1}^{\infty} e^{-ix}) = \exp(2(e^x - 1))^{-1}.$$

Thus, for positive values of x close enough to 0 the inequality

$$\Gamma(x) < k2^k(e^x - 1)^{-k} / \exp(2(e^x - 1))^{-1}$$

holds—which proves (2.10).

The df's $G_i(x)$ defined by (2.5), as well as the $\Gamma_{kn}(x)$, are everywhere continuous. Therefore, by virtue of (2.10), and the uniform convergence in $[a, \infty)$, the limit $\Gamma(x)$ is also continuous (which shows, incidentally, that the convergence of $\Gamma_{kn}(x)$ is uniform in $-\infty < x < \infty$).

The function $\Gamma(x)$ is strictly increasing on $[0, \infty)$ so that the inverse function

$$(2.11) \quad g(x) = \Gamma^{-1}(x)$$

exists and is also increasing and continuous on $[0, 1)$ and satisfies the conditions

$$(2.12) \quad g(0) = 0, \quad g(1) = +\infty.$$

Now let $\Phi(x)$ be an arbitrary df and consider a sequence of functions $F_i(x)$ defined as follows:

$$(2.13) \quad \begin{aligned} F_i(x) &= G_i(g(\Phi(x))) && \text{if } \Phi(x) < 1, \\ &= 1 && \text{if } \Phi(x) = 1; \end{aligned}$$

where the $G_i(x)$ are as in (2.5). Due to (2.12), the $F_i(x)$ are df's.

Let ξ_i be a sequence of mutually independent rv's such that $P(\xi_i \leq x) = F_i(x)$. It follows from (2.5) and (2.13) that $P(\xi_i \leq x) = P(\eta_i \leq g(\Phi(x)))$, therefore we get for any k and n ($1 \leq k \leq n$) also $P(\xi_{kn} \leq x) = P(\eta_{kn} \leq g(\Phi(x)))$ which by (2.6) can be written in the form

$$\Phi_{kn}(x) = \Gamma_{kn}(g(\Phi(x))).$$

Hence by (2.9) and (2.11) we conclude that (2.4) holds. This proves our theorem

Let us remark that the theorem becomes trivial in the case $k = 1$: for a given df $\Phi(x)$ we can take $F_i(x) = \Phi^{2^{-i}}(x)$. However, in the general case we have not been able to invent such a simple construction for the $F_i(x)$.

The theorem just proved shows that, if apart from the mutual independence no other restriction is imposed on the rv's ξ_i , then any df $\Phi(x)$ may be considered as a limit law for the k th term of some variational series. However, it is natural to require that the initial suitably normalized rv's ξ_i should—in some sense—be individually negligible in the limit, so that the role of a single component participating in the formation of the variable ξ_{kn} becomes vanishingly small as $n \rightarrow \infty$. Keeping the above notations, let us introduce the following definition:

We will say that the df $\Phi(x)$ belongs to the class G_k if there exist a sequence

of df's $F_i(x)$ and real constants $a_n > 0$ and b_n such that

$$(2.14) \quad \lim_{n \rightarrow \infty} \Phi_{kn}(a_n x + b_n) = \Phi(x)$$

at each point of continuity of the function $\Phi(x)$, and such that for every x , for which $\Phi(x) > 0$,

$$(2.15) \quad \lim_{n \rightarrow \infty} F_i(a_n x + b_n) = 1$$

uniformly in i ($1 \leq i \leq n$).

(Let us remark that in the case when the rv's ξ_i are identically distributed then (2.15) is contained in (2.14).)

Our aim is to give an exact description of the class G_k .

For sake of brevity we will use the following notations

$$*_\Phi = \inf \{x : \Phi(x) > 0\}, \quad \Phi_* = \sup \{x : \Phi(x) < 1\}.$$

If $\Phi(x)$ is a df, then $*_\Phi(\Phi_*)$ will be called its *left (right) end*.

Since each improper df trivially belongs to G_k , the limit distributions $\Phi(x)$ are assumed to be proper, i.e. $*_\Phi < \Phi_*$.

Finally, let us note that as a consequence of the weak convergence required in (2.14), every non-decreasing function $\Phi(x)$ which satisfies the conditions $\Phi(-\infty) = 0, \Phi(+\infty) = 1$ is a df, and equality of two df's means equality at their points of continuity.

3. The class G_1 . Let P be the class of all df's $\Phi(x)$ that have the following property: for every $\beta > 0$ there exists a non-decreasing function $\phi_\beta(x)$, such that for all x

$$(3.1) \quad \Phi(x) = \Phi(x + \beta)\phi_\beta(x).$$

Let Q be the class of all df's $\Phi(x)$ that have the following properties:

$$(3.2) \quad \Phi_* < \infty$$

and for every α ($0 < \alpha < 1$) there exists a non-decreasing function $\phi_\alpha(x)$, such that for all x

$$(3.3) \quad \Phi(x + \Phi_*) = \Phi(\alpha x + \Phi_*)\phi_\alpha(x)$$

(observe that $P \cap Q \neq \emptyset$).

Let R be the set of all $\Phi(x) \in Q$, which are continuous at the point $x = \Phi_*$.

The case $k = 1$ was studied in [7]–[10] under somewhat more restrictive requirements: it was assumed that a df is—by definition—continuous from the left and that the convergence in (2.4) holds at every point. With these assumptions, the class of the limit distributions was called *class G* and it was proved [10] that $G = P \cup R$. In the present—more general—situation, we get the following.

THEOREM 3.1.

$$G_1 = P \cup Q.$$

PROOF. By [7] it is sufficient to prove that $Q \subset G_1$. So let us suppose that the df $\Phi(x)$ belongs to Q . Without loss of generality we can assume, because of (3.2), that $\Phi_* = 0$. Let us first assume that ${}^*\Phi > -\infty$. Then, by (3.3) [the function

$$(3.4) \quad \begin{aligned} H(x; \alpha) &= 0, & \text{if } x < {}^*\Phi, \\ &= \Phi(x)/\Phi(\alpha x), & \text{if } x > {}^*\Phi, \end{aligned}$$

is a df for every fixed α ($0 < \alpha < 1$). Define

$$\begin{aligned} D(x) &= 0 & \text{if } x < -1/e, \\ &= 1 + 1/\ln |x| & \text{if } -1/e < x < 0, \\ &= 1 & \text{if } x > 0, \end{aligned}$$

and

$$\Phi(0-) = a \quad (0 < a \leq 1).$$

We define the desired sequence $F_i(x)$ by

$$(3.5) \quad F_i(x) = H((i + 1)x; \alpha_i)D^{t_i}(x), \quad i = 1, 2, \dots,$$

where $\alpha_i = i/(i + 1)$, $t_i = (\ln \alpha_i) \ln a$. Taking $a_n = 1/(n + 1)$, $b_n = 0$, we verify that for $x > {}^*\Phi$

$$\prod_{i=1}^n D^{t_i}(a_n x) \rightarrow a \quad (n \rightarrow \infty)$$

and

$$\prod_{i=1}^n H((i + 1)a_n x; \alpha_i) \rightarrow \Phi(x)/a \quad (n \rightarrow \infty),$$

since

$$\prod_{i=1}^n H((i + 1)a_n x; \alpha_i) = \Phi(x)/\Phi((n + 1)x).$$

Hence, it follows in virtue of (2.2) and (3.5) that (2.14) holds for all $x > {}^*\Phi$. On the other hand, by (3.4) and (3.5) it is clear that (2.14) holds also for $x < {}^*\Phi$.

It is easy to check that if $x > {}^*\Phi$, then (2.15) is fulfilled by any fixed i and by $i = n$. Hence, since the sequence a_n is monotone, we conclude that (2.15) is fulfilled uniformly in i ($1 \leq i \leq n$). Thus the df $\Phi(x)$ belongs to G_1 .

The case ${}^*\Phi = -\infty$ can be treated in the same way by defining the function $H(x; \alpha)$ by

$$\begin{aligned} H(x; \alpha) &= 0 & \text{if } x < \alpha/(\alpha - 1), \\ &= \Phi(x)/\Phi(\alpha x) & \text{if } x > \alpha/(\alpha - 1). \end{aligned}$$

An example of a df which belongs to G_1 but is discontinuous at its left end (and, therefore, does not belong to G) is given by

$$\begin{aligned} \Phi(x) &= \exp(x - 1) & \text{if } x < 0, \\ &= 1 & \text{if } x > 0. \end{aligned}$$

A more transparent characterization of the class G_1 is contained in

THEOREM 3.2. [9] *The df $\Phi(x)$ belongs to G_1 if and only if it is logarithmically convex ($\Phi(x) \in P$) or the function $\Phi(*\Phi - e^{-x})$ is logarithmically convex ($\Phi(x) \in Q$).*

It follows from this that the only possible points of discontinuity of a df belonging to G_1 are its ends. Moreover, the right end can be a point of discontinuity only if the df belongs to Q .

In the sequel we shall need the following;

LEMMA 3.1. *Let $\Phi(x) \in G_1$ and assume $*\Phi > -\infty$. Then the sequences $F_i(x)$, a_n and b_n that appear in (2.14) and (2.15), can be chosen so, that for every $x < *\Phi$ and k we will have*

$$(3.6) \quad F_{n-s}(a_n x + b_n) = 0, \quad s = 0, 1, \dots, k,$$

for all sufficiently large n .

PROOF. Let $\Phi(x) \in P$. Then by (3.1) the function

$$\begin{aligned} H(x; \beta) &= 0 && \text{if } x < *\Phi, \\ &= \Phi(x)/\Phi(x + \beta) && \text{if } x > *\Phi, \end{aligned}$$

is a df for every $\beta > 0$. Taking

$$\begin{aligned} F_i(x) &= H(x - \sum_{j=1}^i (1/j); 1/i), && i = 1, 2, \dots, \\ a_n &= 1, && b_n = \sum_{j=1}^n (1/j), \end{aligned}$$

we verify the validity of (2.14) and (2.15). On the other hand, for every s ($0 \leq s \leq n$) we have

$$F_{n-s}(a_n x + b_n) = 0 \quad \text{if } x < *\Phi - \sum_{j=n-s+1}^n (1/j).$$

Thus (3.6) holds for $x < *\Phi$ and $n > k/(*\Phi - x) + k - 1$. The case $\Phi(x) \in Q$ may be handled in the same way, by using the sequences that were constructed in course of the proof of Theorem 3.1.

4. The class G_k ($k \geq 1$). The characterization of the class G_k is given by

THEOREM 4.1. *The df $\Phi(x)$ belongs to G_k if and only if it can be represented in the form*

$$(4.1) \quad \Phi(x) = \phi(x) \sum_{s=0}^{k-1} [(-\ln \phi(x))^s / s!],$$

where $\phi(x)$ is a df of G_1 and $*\Phi = *\phi$.

Let δ_{in} and λ_{in} ($1 \leq i \leq n; n = 1, 2, \dots$) be numerical sequences. We shall need the following two lemmas.

LEMMA 4.1. *Let*

$$(4.2) \quad \delta_{in} \geq 0 \quad \text{and} \quad \max_{1 \leq i \leq n} \delta_{in} \rightarrow 0 \quad (n \rightarrow \infty).$$

If $\sum_{i=1}^n \delta_{in} \rightarrow \delta \quad (n \rightarrow \infty)$, then, keeping the notation of Section 2, we have for every s

$$\sum \prod_{(n*s)} \delta_{in} \rightarrow \delta^s / s! \quad (n \rightarrow \infty).$$

For a proof see [5], Section 15.

LEMMA 4.2. *Let*

$$(4.3) \quad 0 < \lambda_{in} \leq 1 \quad \text{and} \quad \min_{1 \leq i \leq n} \lambda_{in} \rightarrow 1 \quad (n \rightarrow \infty).$$

Denote

$$\pi_n = \prod_{i=1}^n \lambda_{in}, \quad \sigma_n = \sum_{i=1}^n (1 - \lambda_{in}), \quad \bar{\sigma}_n = \sum_{i=1}^n (1/\lambda_{in} - 1).$$

(a) *If one of the sequences π_n , σ_n or $\bar{\sigma}_n$ converges (to a finite or infinite limit) then the other two also converge and*

$$(4.4) \quad \lim_{n \rightarrow \infty} \pi_n = \lim_{n \rightarrow \infty} \exp(-\sigma_n) = \lim_{n \rightarrow \infty} (-\bar{\sigma}_n).$$

(b) *Let s be an arbitrary non-negative integer. If*

$$(4.5) \quad \pi_n \rightarrow \pi > 0 \quad (n \rightarrow \infty),$$

then

$$(4.6) \quad \sum \prod_{(n*s)} (1/\lambda_{in} - 1) \rightarrow (-\ln \pi)^s/s! \quad (n \rightarrow \infty),$$

while if

$$(4.7) \quad \pi_n \rightarrow 0 \quad (n \rightarrow \infty),$$

then also

$$(4.8) \quad \pi_n \sum \prod_{(n*s)} (1/\lambda_{in} - 1) \rightarrow 0 \quad (n \rightarrow \infty).$$

PROOF. (a) It is well known that for any $0 < \alpha \leq 1$

$$\exp(1 - 1/\alpha) \leq \alpha \leq \exp(\alpha - 1).$$

Therefore, for every n

$$(4.9) \quad \exp(-\bar{\sigma}_n) \leq \pi_n \leq \exp(-\sigma_n).$$

On the other hand, for arbitrary $\epsilon(0 < \epsilon < 1)$ we will have, by (4.3), for sufficiently large n

$$(4.10) \quad \bar{\sigma}_n \leq \sigma_n/(1 - \epsilon),$$

which, together with (4.9), proves this part of the lemma.

(b) Let us put $\delta_{in} = 1/\lambda_{in} - 1$, then by (4.3) the conditions (4.2) hold and $\sum_{i=1}^n \delta_{in} = \bar{\sigma}_n$. Since by (4.4) and (4.5) $\lim_{n \rightarrow \infty} \bar{\sigma}_n = -\ln \pi$, then according to Lemma 4.1 we get (4.6). Obviously

$$\sum \prod_{(n*s)} (1/\lambda_{in} - 1) \leq \bar{\sigma}_n^s.$$

Hence, putting $\epsilon = \frac{1}{2}$ in (4.10) we have for sufficiently large n

$$\sum \prod_{(n*s)} (1/\lambda_{in} - 1) \leq (2\sigma_n)^s.$$

Finally, using (4.9), we see that for large n

$$(4.11) \quad \pi_n \sum \prod_{(n*s)} (1/\lambda_{in} - 1) \leq (2\sigma_n)^s \exp(-\sigma_n).$$

Suppose now that (4.7) holds. Then in virtue of (4.4) $\sigma_n \rightarrow \infty$ ($n \rightarrow \infty$). Since for any positive s we have

$$x^s \exp(-x) \rightarrow 0 \quad (x \rightarrow \infty),$$

then (4.8) follows immediately from (4.11).

Notice that both lemmas obviously remain valid if the sequence of natural indices n is replaced by any subsequence n' .

PROOF OF THEOREM 4.1. *Necessity.* Let $\Phi(x) \in G_k$. Consequently there exist $F_i(x)$, a_n and b_n such that (2.14) and (2.15) hold. Consider the sequence of the df's of the corresponding maximal terms and let $\phi(x)$ be any of its partial limits, i.e. $\Phi_{1n'}(a_n x + b_n) \rightarrow \phi(x)$ ($n' \rightarrow \infty$).

We first prove

$$(4.12) \quad * \phi = * \Phi.$$

Since for every $k \geq 1$ we have $\Phi_{1n}(x) \leq \Phi_{kn}(x)$, then clearly

$$(4.13) \quad * \phi \geq * \Phi.$$

Let $x > * \Phi$ be an arbitrary fixed number. For given i and n ($1 \leq i \leq n$) denote

$$\lambda_{in} = F_i(a_n x + b_n),$$

then by (2.15) our sequence λ_{in} satisfies all of the hypotheses of Lemma 4.2 for sufficiently large n . On the other hand, from some n on we can use the expression (2.3). Therefore, introducing the notations of Lemma 4.2, we have by (2.2)

$$(4.14) \quad \Phi_{kn}(a_n x + b_n) = \pi_n \sum_{s=0}^{k-1} \sum \prod_{(n*s)} (1/\lambda_{in} - 1).$$

Hence, should we assume

$$\pi_{n'} \rightarrow \phi(x) = 0 \quad (n' \rightarrow \infty)$$

we would get according to Lemma 4.2 that $\Phi(x) = 0$ too, which is impossible, since $x > * \Phi$. Thus $* \phi \leq * \Phi$ and by (4.13) equality (4.12) is proved.

Now let x ($x > * \Phi$) be a point of continuity of the function $\Phi(x)$. By (4.12)

$$\phi(x) = \lim_{n' \rightarrow \infty} \pi_{n'} > 0.$$

Therefore, according to Lemma 4.2, and (4.14), the df $\Phi(x)$ has the form (4.1).

It is easy to verify that for any $k \geq 1$ the function

$$(4.15) \quad \psi(x) = x \sum_{s=0}^{k-1} [(-\ln x)^s / s!], \quad \psi(0) = 0,$$

is strictly increasing in $[0, 1]$. Hence the representation of a df $\Phi(x)$ by means of a non-decreasing function $\phi(x)$ in the form (4.1) is unique. Thus we conclude that $\phi(x)$ is a df and

$$(4.16) \quad \phi(x) = \lim_{n \rightarrow \infty} \Phi_{1n}(a_n x + b_n).$$

Since (2.15) holds for all $x > * \phi$, then $\phi(x) \in G_1$.

Sufficiency. Let the df $\Phi(x)$ have the form (4.1). Then there exist $F_i(x)$, a_n

and b_n such that (4.16) holds and so does (2.15) for $x > *ϕ$. Consider the sequence $Φ_{kn}(a_n x + b_n)$. It clearly follows from the arguments used in the first part of the proof that this sequence converges to $Φ(x)$ if $x > *ϕ$. Therefore, the proof will be complete if we show that

$$(4.17) \quad Φ_{kn}(a_n x + b_n) \rightarrow 0 \quad (n \rightarrow \infty)$$

for $x < *ϕ$. For these values of x we shall make use of the expression (2.1).

It is clear that if $1 \leq s \leq k - 1$, then in each collection of indices of the form $(n * s)^c$ which consists of $n - s$ different indices, there exists at least one index i' such that $n - k \leq i' \leq n$. On the other hand, according to Lemma 3.1, there is no loss of generality if we assume that for $x < *ϕ$ and sufficiently large n equalities (3.6) hold. Therefore, if $x < *ϕ$ then from some n on all the terms of the sum in (2.1) vanish, i.e. we get (4.17) and the theorem was proved.

In the course of the proof we saw that if for some $k > 1$ and some $F_i(x)$, a_n and b_n (2.14) and (2.15) hold, then—for the same $F_i(x)$, a_n and b_n —the sequence $Φ_{1n}(a_n x + b_n)$ converges too, i.e. (4.16), the left ends of both limits $Φ(x)$ and $ϕ(x)$ coincide and the equality (4.1) holds. However, if (4.16) holds and (2.15) is fulfilled for all $x > *ϕ$, then for $k > 1$ the convergence of the sequence $Φ_{kn}(a_n x + b_n)$ and the equality (4.1) are guaranteed only in the interval $(*ϕ, \infty)$, and we assert nothing concerning the behavior of the sequence in $(-\infty, *ϕ)$. (It is for this reason that we needed Lemma 3.1.) However, the following proposition can be easily established.

THEOREM 4.2. *If (2.14) and (2.15) are satisfied for some k and some $F_i(x)$, a_n and b_n and the left end of the limiting function is a point of continuity (or it is $-\infty$), then (2.14) and (2.15) are satisfied for each k , the left end of the limiting law is independent of k and (4.1) holds.*

REMARK 4.1. Now let the mutually independent rv's $ξ_i$ have the same df $F(x)$, then

$$(4.18) \quad Φ_{kn}(x) = \sum_{s=0}^{k-1} \binom{n}{s} F^s(x) (1 - F(x))^{n-s}.$$

From Gnedenko's result concerning the class of the limit distributions for the maximal term $ξ_{1n}$ we obtain immediately Smirnov's laws (1.1), by using Theorem 4.2, since $Φ_α(x; 1)$, $Ψ_α(x; 1)$ and $Λ(x; 1)$ are everywhere continuous.

The class of df's $F(x)$ for which constants a_n and b_n may be found, such that (2.14) holds and $Φ_{kn}(x)$ is given by (4.18), is called the domain of attraction of the law $Φ(x)$.

The domains of attraction of the df's $Φ_α(x; 1)$, $Ψ_α(x; 1)$ and $Λ(x; 1)$ were first studied by R. de Mises [11]. A complete solution of this problem was given by Gnedenko [3]. Another characterization of the domain of attraction of the law $Λ(x; 1)$ was given in [6].

Smirnov showed [12] that the domain of attraction of any df from (1.1) does not depend on k , i.e. it coincides with the domain of attraction of the corresponding df that is obtained by putting $k = 1$. It is easy to see that also this result of Smirnov is an immediate corollary of our Theorem 4.2.

REMARK 4.2. The expression (4.1) admits also of another interpretation. Consider normalized rv's

$$\xi'_{in} = (\xi_i - b_n)/a_n, \quad i = 1, \dots, n,$$

where ξ_i are mutually independent and $a_n > 0$, b_n are real numbers. For given x and n let us denote by $\eta_n(x)$ the number of ξ'_{in} whose values occur in the interval (x, ∞) , and let

$$P_{kn}(x) = P(\eta_n(x) = k).$$

Then

$$\begin{aligned} P_{kn}(x) &= \Phi_{1n}(x) && \text{if } k = 0, \\ (4.19) \quad &= \Phi_{k+1,n}(x) - \Phi_{kn}(x) && \text{if } 1 \leq k \leq n - 1, \\ &= 1 - \Phi_{nn}(x) && \text{if } k = n. \end{aligned}$$

Now let ξ_i , a_n and b_n be such that the limit (4.16) exists and (2.15) hold for $x > * \phi$. Then it follows from (4.1) and (4.19) that for constant $k \geq 0$ and $x > * \phi$ we have

$$P_{kn}(x) \rightarrow \phi(x) (-\ln \phi(x))^k / k! \quad (n \rightarrow \infty).$$

Thus the distribution of the rv $\eta_n(x)$ converges to the Poisson distribution whose parameter is equal to $-\ln \phi(x)$. In particular, if $\phi(x) = e^x$ ($x < 0$) and $x_1 < x_2 < 0$, then the expected number of ξ'_{in} which occur in the interval (x_1, x_2) asymptotically equals the length of the interval.

We conclude with

THEOREM 4.3. For every k we have $G_k \subset G_1$. In particular, if $\phi(x) \in P(Q)$ then also $\Phi(x) \in P(Q)$, where $\Phi(x)$ is defined by (4.1).

PROOF. Let $\psi(x)$ be a non-decreasing function in $[0, 1]$, $0 \leq \psi(x) \leq 1$. If $\psi(e^x)$ and $\phi(x)$ ($0 \leq \phi(x) \leq 1$) are logarithmically convex, then so is the function $\psi(\phi(x))$.

Indeed, for any non-positive x and y we have

$$\psi(e^x)\psi(e^y) \leq \psi^2(\exp((x + y)/2)),$$

hence

$$\psi(\phi(x))\psi(\phi(y)) \leq \psi^2((\phi(x)\phi(y))^{1/2}).$$

But since

$$\phi(x)\phi(y) \leq \phi^2((x + y)/2)$$

and $\psi(x)$ is non-decreasing then

$$\psi(\phi(x))\psi(\phi(y)) \leq \psi^2(\phi((x + y)/2)).$$

By straightforward differentiation we verify that the function $\psi(x)$ given by (4.15) possesses all the properties formulated above. Thus, by what has just been proved, the theorem follows from Theorems 3.2 and 4.1.

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